Zeta Function - Real

Definition 1. *The **zeta function*** $\zeta(x) \equiv \sum_{n=1}^{\infty} \frac{1}{n^x}$

converges when $x > 1$.

Theorem 1. ***Product Formula***

$$\textit{When } x > 1, \quad \sum_{n=1}^{\infty} \frac{1}{n^x} = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-x}}$$

Proof.

$$\sum_{n=1}^{\infty} n^{-x} = 1 + 2^{-x} + 3^{-x} + 4^{-x} + \dots$$

$$-\left[2^{-x} \sum_{n=1}^{\infty} n^{-x} = 2^{-x} + 4^{-x} + 6^{-x} + 8^{-x} + \dots\right] \qquad \text{multiply and subtract}$$

$$(1 - 2^{-x}) \sum_{n=1}^{\infty} n^{-x} = 1 + 3^{-x} + 5^{-x} + 7^{-x} + 9^{-x} + \dots \qquad \text{prime sieve}$$

$$(1 - 3^{-x})(1 - 2^{-x}) \sum_{n=1}^{\infty} n^{-x} = 1 + 5^{-x} + 7^{-x} + 11^{-x} + 13^{-x} + \dots$$

$$(\dots)(1 - 5^{-x})(1 - 3^{-x})(1 - 2^{-x}) \sum_{n=1}^{\infty} n^{-x} = 1$$

$$\prod_{n=1}^{\infty} \left(1 - p_n^{-x}\right) \sum_{n=1}^{\infty} n^{-x} = 1$$

$$\sum_{n=1}^{\infty} n^{-x} = \prod_{n=1}^{\infty} \left(1 - p_n^{-x}\right)^{-1}$$

□

Zeta Equivalences

$$\mu(n) = \begin{cases} 1 & n \text{ square-free with an even number of prime factors} \\ -1 & n \text{ square-free with an odd number of prime factors} \\ 0 & n \text{ has a squared prime factor} \end{cases} \tag{1}$$

$$\frac{1}{\zeta(x)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^x} \tag{2}$$

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