Series

The sequence of partial sums of an infinite series either converges to a single fixed value or it diverges. Divergent series either oscillate or increase (decrease) without bound.

Harmonic Series

Definition 1. *The **harmonic series** is defined as:* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \;=\; 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

Theorem 1. *The harmonic series diverges.*

Proof.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \\
&> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots \\
&> 1 + \frac{1}{2} + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \dots
\end{aligned}$$

Right side divergence implies left side divergence. □

The ***general harmonic series*** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{an+b}$ also diverges by the limit comparison test.

Geometric Series

Definition 2. *The **geometric series** is defined as:* $\displaystyle\sum_{n=0}^{\infty} ar^n \;=\; a + ar + ar^2 + ar^3 + \dots$

Theorem 2. *The geometric series* $\displaystyle\sum_{n=0}^{\infty} ar^n$ *converges to* $\dfrac{a}{1-r}$ *when* $|r| < 1$.

Proof.

$$\begin{aligned}
\sum_{n=1}^{k} ar^n &= a + ar + ar^2 \cdots + ar^k \\
r\sum_{n=1}^{k} ar^n &= ar + ar^2 + ar^3 \cdots + ar^{k+1} \\
(1-r)\sum_{n=1}^{k} ar^n &= a - ar^{k+1} \\
\sum_{n=1}^{k} ar^n &= \frac{a - ar^{k+1}}{1-r} \\
\lim_{k\to\infty}\sum_{n=1}^{k} ar^n &= \frac{a}{1-r} \quad \text{when} \quad |r| < 1
\end{aligned}$$

□

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