Complex Analysis

Complex Analysis Definitions

$z \in \mathbb{C}$, $\epsilon, \delta \in \mathbb{R}$, $S \subseteq \mathbb{C}$ and $f : S \to \mathbb{C}$.

ϵ-neighborhood: The ϵ-neighborhood of z_0, $N_\epsilon(z_0) \equiv \{z : |z - z_0| < \epsilon\}$.

Interior Point: z is an interior point of S means that $\exists(\epsilon > 0)(N_\epsilon(z) \subseteq S)$.

Open Set: S is open iff every element z of S has some ϵ-neigborhood that is a subset of S.

$$S \text{ is open } \equiv \forall(z \in S)\ \exists(\epsilon > 0)(N_\epsilon(z) \subseteq S)$$

Alternatively, S is open iff every point of S is an interior point.

Relatively Open Set: $V \subseteq S$ is relatively open in S means that $\forall(z \in V)\ \exists(\epsilon > 0)(\big(N_\epsilon(z) \cap S\big) \subseteq V)$.

Set Complement: The complement of a set S, $\mathbb{C} \setminus S \ \equiv\ \{z : z \notin S\}$.

Closed Set: A set S is closed iff its complement is open. Alternatively, a set S is closed iff it contains all its limit points. Note that S could be neither open nor closed.

Limit Point: z_0 is a limit point of a set S iff every ϵ-neigborhood of z_0 contains a point of S other than z_0.

$$z_0 \text{ is a limit point of } S \ \equiv\ \forall(\epsilon > 0)\ \exists(z \in S)(z \in N_\epsilon(z_0)\ \wedge\ z \neq z_0)$$

Isolated Point: Any point in S that is not a limit point is said to be an isolated point.

Function Limit: Given z_0 a limit point of S,

$$\lim_{z \to z_0} f(z) = l \ \equiv\ \forall(\epsilon > 0)\ \exists(\delta > 0)\ \forall(z \in S)((0 < |z - z_0| < \delta) \Rightarrow \big(|f(z) - l| < \epsilon)\big)$$

Function Continuity: The function f is continuous at z_0 iff

$$\begin{aligned} \lim_{z \to z_0} f(z) = f(z_0) &\quad \text{when } z_0 \text{ is a limit point,} \\ \forall(\epsilon > 0)\ \exists(\delta > 0)((N_\delta(z_0) \cap S) \subseteq N_\epsilon\big(f(z_0)\big)) &\quad \text{when } z_0 \text{ is an isolated point.} \end{aligned}$$

Infinite Sequence Limt: The infinite sequence $\{z_n\}$ converges to $l \in \mathbb{C}$ by definition iff for every ϵ-neigborhood of l, there is an $m \in \mathbb{N}$ such that if $n > m$ then z_n is in the given neighborhood.

$$\lim_{z \to \infty} f(z_n) = l \quad \equiv \quad \forall(\epsilon > 0)\ \exists(m)\Big((n > m) \Rightarrow (|z_n - l| < \epsilon)\Big)$$

Proper Convergence: The infinite sequence $\{z_n\}$ converges properly to $l \in \mathbb{C}$ iff $\{z_n\}$ converges to l and $z_n \neq l$ ever.

Complex Derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

Complex Analysis Examples

$$\begin{aligned} \text{Given the sequence } z_n = \frac{i}{n}\Big(1 + (-1)^n\Big) = \Big\{0, \frac{i}{1}, 0, \frac{i}{2}, 0, \frac{i}{2}, \dots\Big\}, \\ \lim_{n \to \infty} z_n = 0 \end{aligned} \tag{1}$$

Complex Analysis Theorems

Closed Sets: A set is closed iff it contains all its limit points.

Function Continuity and Open Sets: A complex function $f : S \to \mathbb{C}$ is continuous iff for every open set U, the inverse image $f^{-1}(U)$ is relatively open in S.

Complex Sequence Convergence: Given a sequence $\{z_n\}$ where $z_n = x_n + iy_n$,

$$\lim_{n \to \infty} z_n = z \quad \Leftrightarrow \quad \Big(\lim_{n \to \infty} x_n = x\ \wedge\ \lim_{n \to \infty} y_n = y\Big)$$

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