

# Calculus

## Function Limit

$$f: \mathbb{R}^1 \rightarrow \mathbb{R}^1$$

$$\begin{aligned} \lim_{x \rightarrow c} f(x) = l \\ \equiv \forall(\epsilon > 0) \exists(\delta > 0) \forall x((0 < |x - c| < \delta) \Rightarrow (|f(x) - l| < \epsilon)) \end{aligned}$$

$$f: \mathbb{R}^n \rightarrow \mathbb{R}^1$$

$$\begin{aligned} \lim_{\mathbf{x} \rightarrow \mathbf{x}_0} f(\mathbf{x}) = l \\ \equiv \forall(\epsilon > 0) \exists(\delta > 0) \forall \mathbf{x}((0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta) \Rightarrow (|f(\mathbf{x}) - l| < \epsilon)) \end{aligned}$$

## Limits

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \tag{1}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 \tag{2}$$

## Differentiation

$$f: \mathbb{R}^1 \rightarrow \mathbb{R}^1, f'(x) \in \mathbb{R}^1$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

## Derivatives

$$\sin' = \cos \tag{3}$$

## Differentiation Formulae

$$(fg)' = f'g + fg' \tag{4}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \tag{5}$$

$$(f(g))' = f'(g)g' \tag{6}$$

## Integration

### Change of Variables

$$\int_a^b f(g)g' dx = \int_{g(a)}^{g(b)} f(u) du \tag{7}$$

$$\iint_D f(x, y) dx dy = \iint_{D'} f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv \tag{8}$$

$$\text{where } \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

### Integration by Parts

$$\int_a^b f(x) dx = \int_{v(a)}^{v(b)} u dv = uv - \int_{u(a)}^{u(b)} v du \tag{9}$$

## Natural Logarithm

$$\ln x \equiv \int_1^x \frac{1}{t} dt$$

$$\int_1^e \frac{1}{t} dt \equiv 1$$

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \rightarrow 0} (1+x)^{1/x} \tag{10}$$

## Partial Derivative

$$f: \mathbb{R}^3 \rightarrow \mathbb{R}^1, \frac{\partial f}{\partial x} \in \mathbb{R}^1$$

$$\frac{\partial f}{\partial x}(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$

## Gradient

$$\text{scaler field } f: \mathbb{R}^3 \rightarrow \mathbb{R}^1, \nabla f \in \mathbb{R}^3$$

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

$$\text{maximum rate of change} = \|\nabla f\|$$

## Divergence

$$\text{vector field } \mathbf{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \text{div } \mathbf{F} \in \mathbb{R}^1$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

## Curl

$$\text{vector field } \mathbf{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \text{curl } \mathbf{F} \in \mathbb{R}^3$$

$$\begin{aligned} \text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathbf{k} \end{aligned}$$

## Line Integral

$$\int_P \mathbf{F} \cdot d\mathbf{r} = \int_P F_1 dx + F_2 dy + F_3 dz = \int_{P'} \left(F_1 \frac{\partial x}{\partial t} + F_2 \frac{\partial y}{\partial t} + F_3 \frac{\partial z}{\partial t}\right) dt$$

$$\text{if } \mathbf{F} = \nabla g,$$

$$\int_P \mathbf{F} \cdot d\mathbf{r} = \int_{P'} dg$$

## Surface Integral

$$\begin{aligned} \iint_S dS &= \iint_{S'} |\mathbf{r}_u \times \mathbf{r}_v| du dv \\ &= \iint_R \sqrt{1 + f_x^2 + f_y^2} dx dy \end{aligned}$$

## Green's Theorem

in the plane

$$\oint \mathbf{F} \cdot d\mathbf{r} = \oint F_1 dx + F_2 dy = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy$$

## Stoke's Theorem

$$\oint \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$$

## Divergence Theorem

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_V \nabla \cdot \mathbf{F} dV$$

## Conservative Field

1. Vector field  $\mathbf{F}$  is conservative.

$$\int \mathbf{F} \cdot d\mathbf{r} \text{ is path independent, } \oint \mathbf{F} \cdot d\mathbf{r} = 0$$

2. A scaler function  $\varphi$  exists such that  $\mathbf{F} = \nabla \varphi$ .

3.  $\text{Curl } \mathbf{F} = \mathbf{0}$ .

4.  $\mathbf{F} \cdot d\mathbf{r}$  is an exact differential.

$$\mathbf{F} \cdot d\mathbf{r} = \nabla \varphi \cdot d\mathbf{r} = d\varphi$$

## Directional Derivative

$$\mathbf{f}: \mathbb{R}^m \rightarrow \mathbb{R}^n \text{ at } \mathbf{c} \text{ in the direction of } \mathbf{u}, \mathbf{f}'(\mathbf{c}; \mathbf{u}) \in \mathbb{R}^n$$

$$\begin{aligned} \mathbf{f}'(\mathbf{c}; \mathbf{u}) &= \lim_{h \rightarrow 0} \frac{\|\mathbf{f}(\mathbf{c} + h\mathbf{u}) - \mathbf{f}(\mathbf{c})\|}{h} \\ &= \nabla \mathbf{f}(\mathbf{c}) \cdot \frac{\mathbf{u}}{\|\mathbf{u}\|} \quad \text{if } \mathbf{f} \text{ is differentiable} \end{aligned}$$

## Total Derivative

The total derivative of  $\mathbf{f}: \mathbb{R}^n \rightarrow \mathbb{R}^m$  at  $\mathbf{c}$  is  $\mathbf{T}_c(\mathbf{v})$  where

$$\begin{aligned} \mathbf{f}(\mathbf{c} + \mathbf{v}) &= \mathbf{f}(\mathbf{c}) + \mathbf{T}_c(\mathbf{v}) + \|\mathbf{v}\| \mathbf{E}_c(\mathbf{v}) \\ &\text{and } \mathbf{E}_c(\mathbf{v}) \rightarrow \mathbf{0} \text{ as } \mathbf{v} \rightarrow \mathbf{0} \end{aligned}$$

$\mathbf{f}$  differentiable at  $\mathbf{c}$  implies that  $\mathbf{f}$  is continuous at  $\mathbf{c}$  and the directional derivative exists in all directions.

The total derivative of  $f: \mathbb{R}^2 \rightarrow \mathbb{R}^1$  at  $(x, y)$  is  $\left[\frac{\partial f}{\partial x} \frac{\partial f}{\partial y}\right]$  where

$$\begin{aligned} f(x+h, y+k) &= f(x, y) + \left[\frac{\partial f}{\partial x} \frac{\partial f}{\partial y}\right] \begin{bmatrix} h \\ k \end{bmatrix} + \|(h, k)\| E(h, k) \\ &\text{and } E(h, k) \rightarrow 0 \text{ as } (h, k) \rightarrow (0, 0) \end{aligned} \tag{11}$$

$$\text{Note: } \left[\frac{\partial f}{\partial x} \frac{\partial f}{\partial y}\right] \begin{bmatrix} h \\ k \end{bmatrix} = \nabla f(x, y) \cdot (h, k)$$