Proof

Square root of two is irrational

Proof.

$$
\begin{array}{l}
p, q, k \in \mathbb{N} \\
\sqrt{2} \in \mathbb{R} \\
\left[\begin{array}{l}
\sqrt{2} \in \mathbb{Q} \\
\exists (p,q)\big((\sqrt{2} = p/q) \wedge \big(\gcd(p,q) = 1\big)\big) \\
(p/q)^2 = (\sqrt{2})^2 = 2 \ , \ p^2 = 2q^2 \\
2|p^2 \ , \ 2|p \\
\exists (k)(p = 2k) \\
2q^2 = 4k^2 \ , \ q^2 = 2k^2 \\
2|q^2 \ , \ 2|q \\
\big(\gcd(p,q) = 1\big) \wedge \big(2|p \wedge 2|q\big)
\end{array}\right. \\
\sqrt{2} \notin \mathbb{Q} \\
\sqrt{2} \in \mathbb{R} \backslash \mathbb{Q} \quad \square
\end{array}
$$

□

Irrational power of irrational number

An irrational power of an irrational number could give a rational result.

Proof.

$$
\begin{array}{l}
s, t \in \mathbb{R} \backslash \mathbb{Q} \\
\sqrt{2} \in \mathbb{R} \backslash \mathbb{Q} \\
\left[\begin{array}{l}
\sqrt{2}^{\sqrt{2}} \in \mathbb{Q} \\
\exists (s,t)(s^t) \in \mathbb{Q}
\end{array}\right. \\
\left[\begin{array}{l}
\sqrt{2}^{\sqrt{2}} \notin \mathbb{Q} \\
\sqrt{2}^{\sqrt{2}} \in \mathbb{R} \backslash \mathbb{Q} \\
\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2 \in \mathbb{Q} \\
\exists (s,t)(s^t) \in \mathbb{Q}
\end{array}\right. \\
\exists (s,t)(s^t) \in \mathbb{Q} \quad \square
\end{array}
$$

□

Product of sum of squares

If two integers can each be expressed as the sum of two squares, then so can their product.

Proof. Let m and m be integers such that each can be expressed as the sum of two squares:

$$
\begin{aligned}
m &= a^2 + b^2 \\
n &= c^2 + d^2
\end{aligned}
$$

Let z_m and z_n be the complex numbers given by $z_m = a + ib$ and $z_n = c + id$, so we have:

$$
\begin{aligned}
m &= |z_m|^2 = a^2 + b^2 \\
n &= |z_n|^2 = c^2 + d^2
\end{aligned}
$$

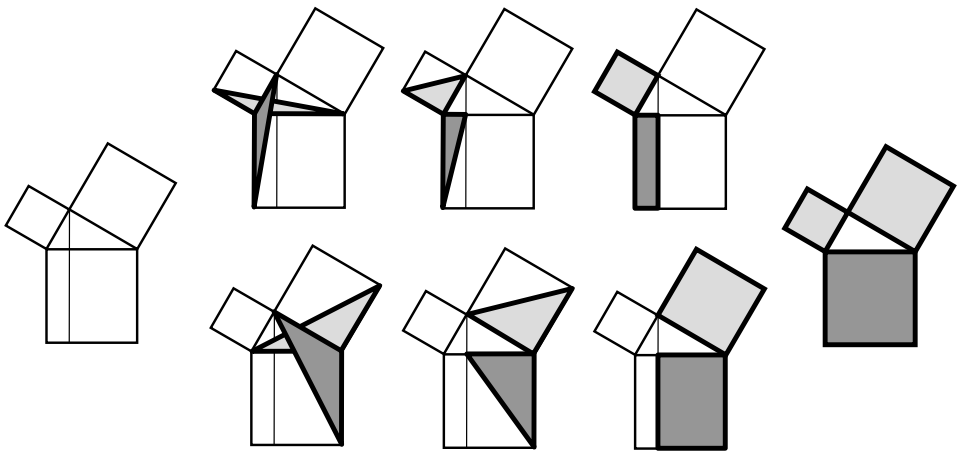
Now we form the product:

$$
\begin{aligned}
mn &= (|z_m|^2)(|z_n|^2) \\
&= (|z_m||z_n|)^2 \\
&= |z_m z_n|^2 \\
&= |(a + ib)(c + id)|^2 \\
&= |(ac - bd) + i(ad + bc)|^2 \\
&= (ac - bd)^2 + (ad + bc)^2
\end{aligned}
$$

Therefore the product mn can always be expressed as the sum of two squares, namely $(ac - bd)^2$ and $(ad + bc)^2$. □

Pythagoras Theorem

Pythagoras Theorem: $c^2 = a^2 + b^2$



□

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